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By J.V. Cox

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A PRELIMINARY STUDY ON FINITE ELEMENT-HOSTED COUPLINGS WITH THE BOUNDARY ELEMENT METHOD

ABSTRACT This study investigates the theoretical and numerical basis of finite element-hosted couplings with the boundary element method (BEM). The emphasis of the study is on (1) conceptually unifying the different coupling approaches and (2) providing an algorithm to facilitate future implementation. Overviews of both the indirect and direct boundary element formulations are included. The theoretical development of BEM stiffness matrices is limited to a physically intuitive derivation based on the indirect BEM. Discontinuity between finite element and boundary element regions is an inherent trait of the boundary element method; this behavior is not eliminated by simply matching the order of shape functions between the two methods. A discussion of the numerical implementation of BEM-formulated stiffness matrices addresses both the direct and indirect BEMs. The direct BEM is better suited in coupled solution approaches due to its simpler traction-displacement relationship. Algorithms are included which outline the calculations unique to obtaining a stiffness matrix from the BEM.

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METRIC CONVERSION FACTORS

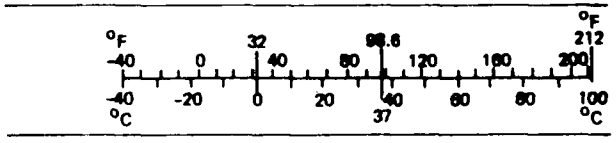
Approximate Conversions to Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
in	inches	*2.5	centimeters	cm
ft	feet	30	centimeters	cm
yd	yards	0.9	*meters*	*m*
mi	miles	1.6	kilometers	km
		AREA		
in^2	square inches	6.5	square centimeters	cm^2
ft^2	square feet	0.09	square meters	m^2
yd^2	square yards	0.8	square meters	m^2
mi^2	*square miles*	*2.6*	*square kilometers*	*km^2*
	acres	0.4	hectares	ha
		MASS (weight)		
oz	ounces	28	grams	g
lb	pounds	0.45	kilograms	kg
	short tons (2,000 lb)	0.9	tonnes	t
		VOLUME		
tsp	teaspoons	5	milliliters	ml
Tbsp	tablespoons	15	milliliters	ml
fl oz	fluid ounces	30	milliliters	ml
c	cups	0.24	*liters*	*l*
pt	pints	0.47	liters	l
qt	quarts	0.95	liters	l
gal	gallons	3.8	liters	l
ft^3	cubic feet	0.03	cubic meters	m^3
yd^3	cubic yards	0.76	cubic meters	m^3
		TEMPERATURE (exact)		
°F	Fahrenheit temperature	5/9 (after subtracting 32)	Celsius temperature	°C

*1 in = 2.54 (exactly). For other exact conversions and more detailed tables, see NBS Misc. Publ. 286, Units of Weights and Measures, Price $2.25, SD Catalog No. C13.10:286.

Approximate Conversions from Metric Measures

Symbol	When You Know	Multiply by	To Find	Symbol
		LENGTH		
mm	millimeters	0.04	inches	in
cm	centimeters	0.4	inches	in
m	meters	3.3	feet	ft
m	meters	1.1	yards	yd
km	kilometers	0.6	miles	mi
		AREA		
cm^2	square centimeters	0.16	square inches	in^2
m^2	square meters	1.2	square yards	yd^2
km^2	square kilometers	0.4	square miles	mi^2
ha	hectares (10,000 m^2)	2.5	acres	
		MASS (weight)		
g	grams	0.035	ounces	oz
kg	kilograms	2.2	pounds	lb
t	tonnes (1,000 kg)	1.1	short tons	
		VOLUME		
ml	milliliters	0.03	fluid ounces	fl oz
l	liters	2.1	pints	pt
l	liters	1.06	quarts	qt
l	liters	0.26	gallons	gal
m^3	cubic meters	35	cubic feet	ft^3
m^3	cubic meters	1.3	cubic yards	yd^3
		TEMPERATURE (exact)		
°C	Celsius temperature	9/5 (then add 32)	Fahrenheit temperature	°F



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This study investigates the theoretical and numerical basis of finite element-hosted couplings with the boundary element method (BEM). The emphasis of the study is on (1) conceptually unifying the different coupling approaches and (2) providing an algorithm to facilitate future implementation. Overviews of both the indirect and direct boundary element formulations are included. The theoretical development of BEM stiffness matrices

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CONTENTS

INTRODUCTION

Boundary element methods (BEM) have become an accepted alternative to domain-based methods (finite elements and finite difference) for many classes of boundary value problems (Refs 1 and 2). The direct (DBEM) and indirect (IBEM) formulations are two well established formulations that are used to solve a variety of boundary value problems in engineering. The key to these formulations is the existence of a fundamental singular solution (i.e., the solution for a point source or point load in an infinite domain -- a free-space Green's function).

This singular solution is jointly the blessing and curse of the methods. Since it is a solution of the governing differential equations, problems defined by a self-adjoint differential operator can be expressed by integral equations. Numerically this means that for many classes of problems BEMs require only the boundary to be discretized. The dimensionality of problems is reduced by one, and thus the methods lend themselves to modeling infinite domains. The reduction in dimensionality reduces the number of degrees of freedom in the approximation; however, the resulting equations are nonsymmetric and fully populated in contrast to the symmetric sparse systems characteristic of domain-based methods. The singular nature of the fundamental solution allows the methods to effectively approximate problems having singular or high-gradient stress fields. This same singular nature complicates the numerical integrations inherent to the methods. The fundamental singular solution is well known for many classes of boundary value problems with homogeneous domains. However, for an arbitrary-inhomogeneous domain the fundamental solution is not known, and additional numerical effort is required (Ref 3). Since the methods are relatively new, they have not been as extensively developed as domain methods for nonlinear applications. They also lack the generality (in terms of extensive continuum and structural element libraries) that commercial finite element computer programs possess.

The finite element method (FEM) has been the most highly developed approximate solution technique for many classes boundary value problems (Ref 39). The most general derivation follows from a weighted residual approximation of the boundary value problem. The error introduced by an approximate solution is orthogonalized with respect to a set of weighting functions to obtain the "best" approximate solution. For the Galerkin weighted residual statement the weighting functions correspond to the basis functions in which the approximate solution is expanded. For problems with a self-adjoint operator, the weak form of the Galerkin statement is equivalent to the variational statement of the problem.

Basis functions in the FEM usually are locally based polynomials for which the integrations are relatively simple. The local nature of these functions leads to a sparse system of equations, but this same attribute hinders its ability to model infinite domains. To compensate for this limitation, special elements (infinite elements) have been developed to model infinite domains which include a shape function based on the form of analytical solutions (Ref 4). This contrasts with the BEM, where rather than using analytical solutions to determine the form of shape functions, the analytical solutions are used directly as weighting functions (Ref 5). The generality of the Galerkin and variational statements allows nonlinearities and domain inhomogeneties to be accommodated. For most structural applications the resulting system of equations is also symmetrical.

Considering the individual strengths of boundary- and domain-based methods, some classes of problems may be most effectively solved by combining the methods. Nonlinear soil-structure interaction and fracture mechanics are two applications where a combined solution approach can potentially provide a more effective analysis. For nonlinear soil-structure interaction, the FEM could be used to model the structure and a region of the soil which is expected to exhibit nonlinear behavior; the BEM could be used to approximate the infinite domain. For fracture mechanics, the FEM might be used to model all of the problem except a region local to the crack tip where the BEM could provide a crack tip "element" capable of representing very high gradients.

Objective

The ultimate objective of this research is to determine whether a combination of boundary element and finite element solution methods could more effectively solve nonlinear structural/geotechnical problems than single method approaches. The immediate emphasis is on the formulation of stiffness matrices for both the indirect and direct boundary element methods (IBEM and DBEM, respectively).

Background

The idea of coupling the FEM and BEM is attributed to Wexler (Ref 6), who used integral equation solutions to constrain FE solutions for infinite domain field problems in the early 1970s. Other early examples of using a combined solution approach dealt with wave phenomena, Chen and Mei (Ref 7) and Shaw (Ref 8). Zienkiewicz et al. (Ref 9) proposed a combined solution approach for statics problems followed by Osias' use of a coupled solution to solve elastostatic problems (Ref 10).

There are numerous approaches to coupling the methods (Ref 11). In this study the main classification is based upon "which method hosts the coupling." For a BEM-hosted coupling the FEM subdomain is treated as a BE region; equilibrium and compatibility are approximately enforced continuously along the interface, analogous to the BEM approach to modeling piece-wise homogeneous problems. The resulting equations are nonsymmetric, and thus this approach only lends itself to problems numerically dominated by the BE subdomain (Refs 11, 12, 13, 14 and 15). For a FEM-hosted coupling the BEM subdomain is treated as one or many finite elements. In this "super-element" approach the resulting BE equations reflect the "equivalent" stiffness of the subdomain and can be directly assembled by a FE program. The term "super-element" reflects the high degree of connectivity which will typically be present for a BE region. A coupled solution approach does not seem to necessitate a displacement-based FE formulation; however due to its commercial success, it appears all the studies of coupled solutions have been based on this formulation.

Within FEM-hosted couplings the next level of classification is the procedure used to obtain the stiffness relationship. The two approaches are: (1) a variational approach where a boundary variational equation is the basis of the relationship, and (2) a direct approach where the BE equations are manipulated into a stiffness form. A key step to both approaches, using either BE formulation, is establishing a relationship between the nodal tractions and the nodal displacements; an exception to this is a variational approach with the IBEM which retains the artificial tractions as unknowns in the system of equations (Refs 11 and 12).

Mei (Ref 16) was apparently the first to formulate a coupled solution based on variational techniques (Ref 13). Other early efforts using a variational approach include: Jeng and Wexler (Ref 17); Shaw (Ref 18); Zienkiewicz, Kelly, and Bettess (Ref 9); and Kelly, Mustoe, and Zienkiewicz (Ref 11). Stationarity of the variational statement necessarily produces a symmetric system, and thus this approach is sometimes grouped with some direct approaches as "symmetric couplings." In the variational approach, a variational statement of the problem is written in a boundary form with the absence of body sources or forces (Refs 11 and 19). For elastostatics the variational statement corresponds to a boundary form of the potential energy functional. There have been many modifications of this functional for various purposes including (1) to relax essential boundary conditions or interface compatibility and (2) to enforce equilibrium (Ref 19 and 20).

With a functional, relationships between boundary values obtained from the integral equations are substituted into the functional, reducing the number of unknowns. For a displacement-based FE, formulation the nodal tractions are usually expressed in terms of the nodal displacements, which requires the inversion of a fully populated coefficient matrix. The previously mentioned exception to this approach (using the IBEM), writes the boundary tractions in terms of the artificial tractions. In this case stationarity is taken with respect to both nodal displacements and artificial boundary tractions instead of nodal displacements alone. The lack of a matrix inversion is paid for by an increase in the number of unknowns and a double integration process.

In contrast to the variational formulations, Brebbia (see e.g., Ref 2, 5, or 21) directly manipulated the BE equations into a stiffness form. The Maxwell-Betti reciprocal theorem proves that a stiffness matrix should be symmetric for a linear elastic domain; however, the direct manipulation does not produce a symmetric system. Brebbia symmetrized the system by averaging the off-diagonal terms based on an error minimization argument using the method of least squares. The resulting stiffness matrix was of the same form as obtained by variational methods using the potential energy functional (Ref 11).

Despite the apparent agreement between the two approaches, agreement between proponents of the approaches is scarce. Kelly et al. (Ref 11) state that the modification to obtain symmetry "apparently has no valid foundation except that the energy approach led to a similar form." Brebbia et al. (Ref 2) later admitted that there is no rationale to justify the error minimization and that the matrix should in fact be nonsymmetric. In addition, he questions the combination of variational principles and integral equations used in the variational approach. His argument is not implying that a nonsymmetric stiffness matrix is physically meaningful for an elastic domain; rather he seems to be indicating that the numerical approximations used by the BEM will inherently produce a nonsymmetric stiffness matrix. Initial efforts indicated that the symmetric matrix provided better results; however, more recent comparisons (Ref 20) using quadratic elements and providing for geometric discontinuities indicate the contrary. Fortunately for cases which have a relatively fine boundary discretization, Roudolphi (Ref 22) has found the degree of nonsymmetry to be insignificant. This is likely to be the case for geotechnical applications.

Another problem which is common to both approaches is failure of the stiffness matrix to satisfy equilibrium. Equilibrium requires the terms in each column corresponding to a given global direction to sum to zero. With the DBEM the system of equations can be augmented by two additional equations which cause the approximate traction distribution to identically satisfy equilibrium (Ref 11). By this approach the equilibrium problem is addressed while establishing the nodal traction-displacement relationship instead of after the stiffness matrix is

formed. Hartman (Ref 23) gives a theoretical discussion on the symmetry and equilibrium problems and provides advice on how to correct them. Tullberg and Belteus (Ref 20) compare the accuracy and convergence characteristics of seven stiffness formulations for two simple two-dimensional problems -- uniaxial tension and bending. The seven stiffness matrices were obtained using the DBEM and included all the variations mentioned above.

In addition to the differences used here to classify formulations, there have been numerous other variations. Some researchers (Ref 19) have used element condensation to reduce the unknowns for the BE super-element to those located on the interface between the two methods. This could be useful in fracture mechanics applications where numerous boundary elements might be used near the crack tip. As previously mentioned, modified functionals have been used to relax compatibility requirements (e.g., when interfacing subdomains with different shape functions).

Variations in the BE formulations themselves introduce additional classification parameters. Though not addressed in this study, for some applications traction discontinuity is an important consideration. Many ways of treating this problem have been reported (Ref 20, 25, 26, and 27). A simple though less rigorous approach is to position the collocation points at the interior of the element. When all of the collocation points associated with an element are internally positioned, the elements are referred to as "discontinuous" or "nonconforming" elements (Ref 28). Though the discontinuous elements add some flexibility to the method in terms of mesh refinement, it is at the expense of increasing the number of equations by up to 100 percent. Discontinuous elements would appear to complicate a coupling algorithm since the collocation points do not correspond to node points.

Relationship to Previous NCEL Work

The primary role of this research of the BEM has been to compliment FEM analysis capabilities (i.e., coupled solution techniques). The BEM is in general a more analytically rigorous technique, and thus the fundamentals of the method have been investigated. As a result of these

efforts the potential of the method as a separate analysis tool has also been observed. Thus the efforts at coupling have emphasized a modular approach where it is assumed that both analysis methods can serve as an individual or a coupled tool.

In NCEL's first study (Ref 29) the IBEM formulation was investigated in one- and two-dimensional elastostatics and numerical results were compared with the DBEM and analytical solutions. A crude element (constant distribution) was used but the potential of the method was evident. This study indicated the need for higher order elements and integration techniques to improve results in the near boundary region.

In NCEL's next study (Ref 15) a coupling of the BE and FE methods was investigated. A BEM-hosted coupling was formulated, approximating compatibility and equilibrium on the interfaces. Qualitatively the coupling was a success, but the simple constant elements used for both methods prevented substantial quantitative evaluation. For nonlinear soil-structure applications where the FEM is used to model the nonlinear region, the FEM should host the coupling. Two areas required further study: (1) accuracy of the BEM formulations, and (2) a FEM-hosted coupling approach.

In the third study (Refs 30 and 31) higher order isoparametric boundary elements were investigated along with their associated integration techniques. Lachat and Watson (Ref 32) adapted the isoparametric element formulation directly from FE technology. However, for the BEM the integrands in the coefficient calculations are singular and thus their integration requires special attention. Though much effort had been devoted to dealing with the integration over the singularity an effective numerical approach to performing the integrations in the near boundary region was lacking. A recursive algorithm which adaptively subdivides the element to maintain consistent accuracy independent of the response point location was developed. The new element formulation has effectively eliminated near boundary error resulting from coarse numerical integration and has provided valuable insight to other errors inherent to the boundary element methods.

Scope

In this study, different approaches to obtaining a FEM-hosted coupling are investigated. The emphasis is on: (1) conceptually unifying the different coupling techniques and (2) providing an algorithm to facilitate future implementation. This preliminary report on the study emphasizes the development of an algorithm for obtaining a stiffness matrix from BEMs. A future report should provide a more detailed theoretical background and numerical results. A physically intuitive development of an IBEM stiffness matrix is included, but the DBEM is also addressed. Overviews of both the DBEM and IBEM formulations are given in the following sections to establish nomenclature and to present the necessary equations for background.

Nomenclature

A list of principal symbols used in the report is included as Appendix A. Additional variables which have a limited scope of use are defined within the text. Modifications (such as the addition of a subscript) are also defined in the text unless listed in Appendix A. Instead of the matrix notation defined, indicial notation will sometimes be used to clarify equations. Indices will always be lower case while identifying subscripts or superscripts will be upper case.

THEORY

This study focuses on the development of a stiffness matrix for a coupled solution approach. Each method of analysis is applied to portions of the domain where it is best suited. For illustration the domain Ω (Figure 1) is divided into two subdomains Ω^F, the FE subdomain, and Ω^B, the BE subdomain. The equations of equilibrium are given by

$$\sigma_{ij,j} + \psi_i = 0 \tag{1a}$$

$$\sigma_{ij} = \sigma_{ji} \tag{1b}$$

and the compatibility equations are given by

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{2}$$

The BE subdomain is assumed to be a linear isotopic homogeneous elastic material. Thus the governing constitutive relations, generalized Hook's law, are given as

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\,\varepsilon_{kk}\,\delta_{ij} \tag{3}$$

where λ and μ are Lame's constants expressed in terms of Young's modulus (E) and Poisson's ratio (ν) as

$$\lambda = \frac{E\,\nu}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)} \tag{4}$$

Alternatively the equilibrium (1), compatibility (2), and constitutive (3) relations can be combined to give a single set of equations in terms of displacement written as

$$\mu\,u_{i,jj} + (\lambda+\mu)\,u_{j,ji} + \psi_i = 0 \tag{5}$$

the Navier equations. The boundary conditions are given by

$$u_i(\underline{x}) = \hat{u}_i(\underline{x}) \qquad \text{on } \Gamma_U \tag{6a}$$

$$t_i(\underline{x}) = \hat{t}_i(\underline{x}) \qquad \text{on } \Gamma_T \tag{6b}$$

where $\hat{u}_i(\underline{x})$ and $\hat{t}_i(\underline{x})$ are prescribed distributions of boundary displacements and tractions, respectively. The simple notation does not imply, that the boundary conditions are mutually exclusive; the fully mixed boundary value problem is addressed.

The key solution to Equation (5) for the more popular BEMs is the fundamental singular solution. This is the solution due to a concentrated load in an infinite space which has the same dimension as the problem to be solved. For plane strain conditions the Kelvin solution expresses the displacement field $u_i(\underline{x})$ due to a unit force $e_k(\underline{\xi})$ in an infinite plane. The indices i, j, and k assume values of 1 or 2, and repeated indices imply summation. The Kelvin solution is given by

$$u_i(\underline{x}) = G_{ik}(\underline{x},\underline{\xi})\, e_k(\underline{\xi}) \tag{7a}$$

where

$$G_{ik}(\underline{x},\underline{\xi}) = C_1\left(C_2\, \delta_{ik}\, \ln r - \frac{y_i\, y_k}{r^2}\right) + A_{ik} \tag{7b}$$

$$C_1 = -\frac{1}{8\pi\mu(1-\nu)}$$

$$C_2 = 3 - 4\nu$$

A_{ik} = arbitrary constant tensor based on zero displacement reference distance

$$y_i = x_i - \xi_i$$

$$r^2 = y_i\, y_i$$

By incorporating Equations (2) and (3), the stress field $\sigma_{ij}(\underline{x})$ is given as

$$\sigma_{ij}(\underline{x}) = T_{ijk}(\underline{x},\underline{\xi})\, e_k(\underline{\xi}) \tag{8a}$$

where

$$T_{ijk}(\underline{x},\underline{\xi}) = \frac{C_3}{r^2}\left[C_4(\delta_{ik}y_j + \delta_{jk}y_i - \delta_{ij}y_k) + \frac{2y_i y_j y_k}{r^2}\right] \tag{8b}$$

$$C_3 = -\frac{1}{4\pi(1-\nu)}$$

$$C_4 = 1 - 2\nu$$

Equilibrium conditions applied at a boundary point, with a unit outward normal $n_i(\underline{x})$, and Equation (8a) combine to give the surface tractions $t_i(\underline{x})$ as

$$t_i(\underline{x}) = F_{ik}(\underline{x},\underline{\xi})\, e_k(\underline{\xi}) \tag{9a}$$

where

$$F_{ik}(\underline{x},\underline{\xi}) = \frac{C_3}{r^2}\left[C_4(n_k y_i - n_i y_k) + \left(C_4 \delta_{ik} + \frac{2y_i y_k}{r^2} \right) y_j n_j\right] \tag{9b}$$

Figures 2 and 3 illustrate the singular behavior of the fundamental solution G_{11} and T_{111} respectively, for a point load applied at the origin of the Cartesian system. The singularity of G is order ln(r), while the singularity of T, obtained from derivatives of G, is order 1/r. The plane strain solution can be converted to the plane stress solution by specifying an effective Poisson ratio $\bar{\nu} = \nu/(1+\nu)$.

The fundamental solution is a key ingredient in formulating integral equations for the indirect and direct BEMs. Integral equations are an equivalent statement of a boundary value problem. Finite difference solutions approximate the differential equations; finite element solutions approximate the stationarity of the variational statement (in some instances); and BEMs approximate the integral equations.

The following subsections provide an overview of both the direct and indirect BEMs and their associated integral equations. These overviews are followed by a derivation of an IBEM stiffness matrix. The last subsection discusses the discontinuous nature of coupled solution techniques--an inherent trait of integral equation methods.

Overview of Direct Boundary Element Method

The direct boundary element method is the most highly developed of all the integral equation methods. It was first applied to elastostatics by Rizzo (Ref 33). Cruse and Rizzo (Ref 34) followed with a solution of

the general transient elasto-dynamic problem. Since these early applications, it has matured numericaliy and the scope of application has significantly broadened.

The integral equations on which the DBEM is based are known as the Somigliana identities. The first identity has been obtained by two different derivations. In both cases the fundamental solution plays a key role. The first derivation uses Betti's reciprocal work theorem in which one system is the actual boundary value problem and the second system corresponds to the fundamental solution (Ref 33). Betti (1872-73) and Somigliana (1885-86) were the first to apply potential methods to elasticity (Ref 19). Thus it is not surprising that Betti's theorem and Somigliana's first identity are the Navier equation counterparts to Green's second and third formula from potential theory.

Somigliana's first identity is given by

$$u_j(\underline{\xi}) = \int_\Gamma [G_{ij}(\underline{x},\underline{\xi})\ t_i(\underline{x}) - F_{ij}(\underline{x},\underline{\xi})\ u_i(\underline{x})]\ d\underline{x}$$

$$+ \int_\Omega G_{ij}(\underline{z},\underline{\xi})\ \hat{\psi}_i(\underline{z})\ d\underline{z} \qquad (10a)$$

($\underline{x} \in \Gamma$ and $\underline{z} \in \Omega$)

in which the displacement field is written in terms of the boundary tractions, boundary displacements, and body forces. The second identity is obtained by combining Equation (10a) with the compatibility (Equation (2)) and the constitutive (Equation (3)) relations giving the stress tensor as

$$\sigma_{jk}(\underline{\xi}) = \int_\Gamma [H_{ijk}(\underline{x},\underline{\xi})\ t_i(\underline{x}) - E_{ijk}(\underline{x},\underline{\xi})\ u_i(\underline{x})]\ d\underline{x}$$

$$+ \int_\Omega H_{ijk}(\underline{z},\underline{\xi})\ \hat{\psi}_i(\underline{z})\ d\underline{z} \qquad (10b)$$

in which the stress field is now expressed in terms of boundary tractions, boundary displacements, and body forces. Since the compatibility relation (Equation (2)) involves derivatives of displacement, the kernel functions H and E have singularities of orders $1/r$ and $1/r^2$, respectively.

The DBEM is formulated by the numerical approximation of Equations (10). Equation (10a) is used to obtain the unknown boundary values, and then both equations allow the calculation of internal responses. The two approximations made to obtain the unknown boundary values are: (1) integrating in a piecewise manner, and (2) solving the equation in a boundary collocation sense. That is, the integration is subdivided over boundary elements and domain cells; and the integral equations are applied to a discrete number of points on the boundary.

The early applications of the method used analytical integrations over elements which could model constant or linear distributions of boundary and domain values. More recent applications apply isoparametric element concepts common to the finite element method (Ref 32). However, unlike the FEM all quantities are interpolated at the same order (i.e., linear boundary elements interpolate geometry, displacements, and tractions linearly). The boundary distribution of any field variable γ on a single element is written as

$$\gamma_i(\eta) = \sum_{\alpha=1}^{N_{EN}} N_\alpha(\eta)\, \gamma_{i\alpha} \tag{11}$$

where α is the node index; N_{EN} is the number of element nodes; η is the normalized local curvalinear coordinate; $\gamma_{i\alpha}$ are nodal values of the field variable; and $N_\alpha(\eta)$ are the appropriate polynomial shape functions for the element. A typical quadratic element ($N_{EN} = 3$) is shown in Figure 4. The shape functions for this element are given by

$$N_1(\eta) = \frac{1}{2}(\eta^2 - \eta) \tag{12a}$$

$$N_2(\eta) = (1 - \eta^2) \tag{12b}$$

$$N_3(\eta) = \frac{1}{2}(\eta^2 + \eta) \tag{12c}$$

The kernal functions in the integral equations are singular and require special treatment when considering the response at points on (Ref 1) or near (Ref 30) the boundary. The corresponding numerical details are omitted here for brevity. While the use of the fundamental solution ensures the satisfaction of the governing differential equations, the boundary collocation ensures violation of the boundary conditions. This point is discussed in a later section with respect to a coupled solution approach.

The piecewise integration and boundary collocation allow Equation (10a) to be approximated by a system of linear equations in terms of the nodal boundary values and body forces as

$$\underset{\sim}{G}\,\underline{t} - \underset{\sim}{F}\,\underline{u} + \underset{\sim}{G}'\,\hat{\underline{\psi}} = \underline{0} \tag{13}$$

(for detail see Reference 1 or 2). The body forces are specified and thus the product with $\underset{\sim}{G}'$ gives a known vector. For a mixed boundary value problem the vectors of nodal tractions and displacements contain known and unknown values. To solve for the unknown nodal values we partition the system as

$$\begin{bmatrix} \underset{\sim}{G}_1 & \underset{\sim}{G}_2 \end{bmatrix} \begin{Bmatrix} \underline{t} \\ \hat{\underline{t}} \end{Bmatrix} - \begin{bmatrix} \underset{\sim}{F}_1 & \underset{\sim}{F}_2 \end{bmatrix} \begin{Bmatrix} \underline{u} \\ \hat{\underline{u}} \end{Bmatrix} + \underset{\sim}{G}'\,\hat{\underline{\psi}} = \underline{0} \tag{14}$$

and then collect the unknown nodal values to give

$$\begin{bmatrix} \underset{\sim}{G}_1 & -\underset{\sim}{F}_1 \end{bmatrix} \begin{Bmatrix} \underline{t} \\ \underline{u} \end{Bmatrix} = \begin{bmatrix} -\underset{\sim}{G}_2 & \underset{\sim}{F}_2 \end{bmatrix} \begin{Bmatrix} \hat{\underline{t}} \\ \hat{\underline{u}} \end{Bmatrix} - \underset{\sim}{G}'\,\hat{\underline{\psi}} \tag{15}$$

The unknown nodal values are obtained by solving the above system. Then Somigliana's identities can be used to obtain the desired internal responses.

Overview of Indirect Boundary Element Method

The indirect boundary element method is probably the second most common integral equation method. Like the direct method it also has its origins in classical potential theory (Ref 35). Single- and double-layer potentials were used in the theory of classical electrodynamics to express boundary value problems as integral equations. The most advanced implementation of the method appears to remain in its application to electromagnetic field problems (Refs 6, 17, and 36). The indirect method appears to have first been applied to elastostatics by Massonnet (Ref 37). The numerical development of IBEM, for elasticity problems, paralleled that of the DBEM until the mid 1970s. In recent years it has not been developed as extensively as the direct method; however, its physically meaningful formulation provides insight to both methods (Ref 31).

The integral equations on which the IBEM is based are a single-layer potential statement of the boundary value problem. The domain Ω is embedded in an infinite plane as shown in Figure 5. For elasticity applications the single-layer source corresponds to a vector of artificial tractions, $P_k(\underline{\xi})$ ($\underline{\xi} \in \Gamma$). In this formulation we seek the boundary distribution of artificial tractions which satisfy the prescribed boundary conditions. These tractions are artificial in the sense that they only exist because the domain Ω has been embedded in an infinite plane. They represent an intermediate step in the formulation and do not correspond to the actual boundary tractions. Artificial tractions and body forces can be expressed as a "continuous" distribution of $e_k(\underline{\xi})$. Thus the displacement, stress and traction fields are given by integrating Equations (7a), (8a), and (9a) respectively as

$$u_i(\underline{x}) = \int_\Gamma G_{ik}(\underline{x},\underline{\xi})\, P_k(\underline{\xi})\, d\underline{\xi} + \int_\Omega G_{ik}(\underline{x},\underline{z})\, \hat{\psi}_k(\underline{z})\, d\underline{z} \qquad (16)$$

$$\sigma_{ij}(\underline{x}) = \int_\Gamma T_{ijk}(\underline{x},\underline{\xi})\, P_k(\underline{\xi})\, d\underline{\xi} + \int_\Omega T_{ijk}(\underline{x},\underline{z})\, \hat{\psi}_k(\underline{z})\, d\underline{z} \qquad (17)$$

$$t_i(\underline{x}) = \int_\Gamma F_{ik}(\underline{x},\underline{\xi})\, P_k(\underline{\xi})\, d\underline{\xi} + \int_\Omega F_{ik}(\underline{x},\underline{z})\, \hat{\psi}_k(\underline{z})\, d\underline{z} \qquad (18)$$

Since the solution is expressed as a superposition of the fundamental solution, the governing differential equations are satisfied over the entire plane including the domain Ω.

To determine the distribution of the artificial tractions we bring the field point $\underline{x}$ to the boundary Γ and enforce the boundary conditions, Equation (6). The resulting integral expressions are given as

$$\hat{u}_i(\underline{x}) = \int_\Gamma G_{ik}(\underline{x},\underline{\xi})\, \bar{P}_k(\underline{\xi})\, d\underline{\xi} + \int_\Omega G_{ik}(\underline{x},\underline{z})\, \hat{\psi}_k(\underline{z})\, d\underline{z} \qquad \underline{x} \in \Gamma_U \qquad (19)$$

$$\hat{t}_i(\underline{x}) = \int_\Gamma F_{ik}(\underline{x},\underline{\xi})\, \bar{P}_k(\underline{\xi})\, d\underline{\xi} + \int_\Omega F_{ik}(\underline{x},\underline{z})\, \hat{\psi}_k(\underline{z})\, d\underline{z} \qquad \underline{x} \in \Gamma_T \qquad (20)$$

Equation (19) is regular upon integration while Equation (20) must be interpreted in a Cauchy principal value sense and is thus written as

$$\hat{t}_i(\underline{x}) = \pm \frac{1}{2}\, \delta_{ik} \bar{P}_k(\underline{x}) + \int_\Gamma F_{ik}(\underline{x},\underline{\xi})\, \bar{P}_k(\underline{\xi})\, d\underline{\xi} + \int_\Omega F_{ik}(\underline{x},\underline{z})\, \hat{\psi}_k(\underline{z})\, d\underline{z} \qquad \underline{x} \in \Gamma_T \qquad (21)$$

A tangent line is assumed through $\underline{x}$ and the sign on the first term depends on the orientation of the element with respect to Ω.

The IBEM is formulated by the numerical approximation of Equations (16) through (21). Equations (19) and (21) are used to determine the unknown artificial tractions, and then Equations (16) through (18) allow the calculation of internal responses. As with the DBEM, the two approximations made to obtain the unknowns are: (1) integrating in a piecewise manner and (2) solving the integral equations in a weighted residual sense (on the boundary). The integration is subdivided over boundary elements and domain cells. Normally the integral equations are

satisfied in a collocation sense; however, Lean et.al (Ref 36) report improved accuracy by a Galerkin approximation. This study is limited to a collocation approximation of the equations. Equations (19) and (21) are then approximated as

$$\hat{\underline{u}} = \underset{\sim}{G}\,\bar{\underline{P}} + \underset{\sim}{G}'\,\hat{\underline{\psi}} \tag{22a}$$

$$\hat{\underline{t}} = \underset{\sim}{F}\,\bar{\underline{P}} + \underset{\sim}{F}'\,\hat{\underline{\psi}} \tag{22b}$$

where $\hat{\underline{u}}$ and $\hat{\underline{t}}$ are values of boundary displacements and tractions at collocation points. For continuous elements the collocation points correspond to element node points. The coefficient matrices $\underset{\sim}{G}$, $\underset{\sim}{G}'$, $\underset{\sim}{F}$ and $\underset{\sim}{F}'$ are obtained by integrations of Equations (19) and (21) with respect to the appropriate shape functions. $\bar{\underline{P}}$ is the vector of unknown nodal artificial traction values, and the last term provides the effect of the body forces at each collocation point. For details see References 1, 2, 31, and 38.

For a mixed boundary value problem Equations (22a) and (22b) are applied to Γ_U and Γ_T, respectively, giving

$$\begin{bmatrix} \hat{\underline{u}} \\ \hat{\underline{t}} \end{bmatrix} = \begin{bmatrix} \underset{\sim}{G} \\ \underset{\sim}{F} \end{bmatrix} \bar{\underline{P}} + \begin{bmatrix} \underset{\sim}{G}' \\ \underset{\sim}{F}' \end{bmatrix} \hat{\underline{\psi}} \tag{23}$$

The nodal artificial tractions are obtained by solving the above system. Then Equations (16) through (18) can be used to obtain the desired internal responses. As when developing the system of equations, boundary values are obtained by letting $\underline{x}$ go to the boundary (as Equations (19) and (21)).

Consider a few of the differences between the two BE formulations. Let N_{CP} be the number of boundary collocation points. Ignoring the integrations associated with body forces, the DBEM (Equation 15) and IBEM (Equation 23) require the calculation of $2(2N_{CP})^2$ and $(2N_{CP})^2$ coefficients, respectively. However, in solving the system of equations, the DBEM yields the unknown boundary values directly while the IBEM yields the artificial traction distribution. To calculate all the unknown boundary values the IBEM requires additional $(2N_{CP})^2$

coefficient calculations. Though the total number of coefficients, and thus integrations, required to obtain the unknown boundary values are equal, the computational effort is not. The coefficient matrices of the DBEM are calculated "simultaneously" and thus the "overhead calculations" (e.g., calculation of the jacobian) associated with the numerical integrations are done once. Since these "overhead calculations" are performed twice by the IBEM, it requires more effort. However, this comparison has assumed the analyst needs all the boundary values. They are required by the DBEM to calculate internal responses by the Somigliana identities; they are not needed by the IBEM for internal response calculation and thus the analyst can be selective.

The calculation of internal responses also differs between the methods. The DBEM obtains internal responses by the Somigliana identities, Equation (10), which integrate the effects of both the boundary tractions and displacements. The IBEM only integrates the effects of artificial tractions, Equations (16) through (18). As in developing the system of equations, though the effort is not doubled for the DBEM, it is considerably more. Another important difference in the internal response calculations is the order of singularity of the kernal functions. For two dimensional problems the methods have the following orders of singularity:

	Displacement	Strain, Stress, Traction
DBEM	$o(\ln r)$ and $o(1/r)$	$o(1/r)$ and $o(1/r^2)$
IBEM	$o(\ln r)$	$o(1/r)$

Thus the DBEM must deal with stronger singularities and correspondingly more difficult integrations. This problem is most severe in the near boundary region (Refs 30 and 31). In fairness, the indirect method is not without its problems in calculating internal responses. Though the integration effort is reduced in the IBEM, often the accuracy is too. A problem that is inherent to the IBEM is associated with loading and geometric discontinuities. In these areas, the artificial tractions experience very high gradients even though boundary conditions may be very well behaved. Thus unless these areas receive special treatment in the numerical formulation or modeling, accuracy is locally very poor.

For coupled solution approaches, particularly for infinite domain problems, it is not easy to determine which BEM would be the most effective. Both methods must calculate two coefficient matrices which involve the same orders of singularity. In addition, discontinuities can often be avoided, eliminating them as a factor. In the following section the IBEM is used to illustrate a direct derivation of a stiffness matrix. The ideas can be applied to the DBEM but are explained with the IBEM due to its conceptual simplicity.

A Direct Derivation of an IBEM Stiffness Matrix

This section presents a physically intuitive derivation of an IBEM stiffness matrix. The relation between nodal displacements and tractions is identical to the relation obtained directly by Kelly et al. (Ref 11). The physical approach provides additional insight to the stiffness formulation.

A stiffness matrix relates nodal displacements to generalized nodal forces in the form

$$\underset{\sim}{K}\,\bar{\underline{u}} = \hat{\underline{f}} \tag{24}$$

where $\underset{\sim}{K}$, $\bar{\underline{u}}$ and $\hat{\underline{f}}$ are the stiffness matrix, displacement vector, and generalized force vector, respectively. This basic definition and a numerical solution provided by the IBEM allow a stiffness matrix to be calculated for a BEM region. Consider the displacement boundary value problem shown in Figure 6. The boundary, Γ, is subdivided into n_E boundary elements. The elements are shown as isoparametric quadratic elements for illustration. All displacements are zero except the j^{th} d o f which has a unit displacement. We can now use the IBEM to solve the displacement boundary value problem. As given by Equation (22a) with body forces omitted, the unknown nodal artificial boundary tractions are related to the known nodal displacements by

$$\hat{\underline{u}} = \underset{\sim}{G}\,\bar{\underline{P}} \tag{25}$$

This system of equations can be solved for the nodal artificial traction values which can then be used to obtain the unknown real boundary tractions by an approximation of Equation (18). The nodal tractions can be expressed in matrix form as

$$\bar{t} = \underset{\sim}{F}\,\hat{P} \tag{26}$$

The distribution of the traction along the boundary is then approximated in terms of the locally based shape functions* as

$$\underline{t}(\underline{x}) = \underset{\sim}{M}(\underline{x})\,\bar{t} \tag{27}$$

which by incorporating Equation (26) becomes

$$\underline{t}(\underline{x}) = \underset{\sim}{M}(\underline{x})\,\underset{\sim}{F}\,\hat{P} \tag{28}$$

With the distribution of boundary tractions, generalized nodal forces (on the boundary) can now be obtained. The nodal forces are determined by "measuring" the work done by the tractions due to a series of virtual displacements - displacement shape functions*. The generalized forces are given by

$$\bar{f} = \int_{\Gamma} \underset{\sim}{N}^{T}(\underline{x})\,\underline{t}(\underline{x})\,d\underline{x} \tag{29}$$

For the prescribed displacements $\hat{\underline{u}}$, the generalized forces (Equation (29)) are the j^{th} column of a stiffness matrix for the domain Ω. To completely calculate the stiffness matrix we must individually subject each remaining nodal degree of freedom to a unit perturbation, solve the corresponding displacement boundary value problem, and then determine the generalized forces by Equation (29) to obtain the corresponding column of $\underset{\sim}{K}$.

We now seek to express the above procedure in equation form. Using indicial notation, the nodal displacement vector for the J^{th} perturbation is given by

*In this context the shape functions are locally based but defined globally. That is, they are nonzero on one or two elements but defined on all Γ.

$$\bar{u}_i^J = \begin{cases} 1, & i = J \\ 0, & i \neq J \end{cases} \tag{30}$$

where i ranges from 1 to the number of degrees of freedom (n). Each $\underline{\bar{u}}^J$ is the J^{th} coordinate vector in the n dimensional space. Combining all of the vectors into a single matrix (i.e., the superscript J becomes an index j) allows Equation (25) to be rewritten as

$$\bar{u}_{ij} = G_{ik} \bar{P}_{kj} \tag{31}$$

By Equation (30), $\bar{u}_{ij}$ is simply the identity matrix. Thus we can easily solve for the artificial tractions as

$$\bar{P}_{ij} = G_{ij}^{-1} \tag{32}$$

This formulation provides insight to the character of $\underset{\sim}{G}^{-1}$; each column of $\underset{\sim}{G}^{-1}$ is the vector of artificial tractions resulting from a unit perturbation of the corresponding d o f. $\underset{\sim}{G}$ is of full column rank and thus invertable if the collocation point positions are unique. Inserting this result into Equations (28) and (29) yields the stiffness matrix as

$$K_{ij} = \int_\Gamma N_{ki}(\underline{x})\, M_{kl}(\underline{x})\, d\underline{x}\, F_{lm}\, G_{mj}^{-1} \tag{33}$$

or in matrix notation

$$\underset{\sim}{K} = \int_\Gamma \underset{\sim}{N}^T(\underline{x})\, \underset{\sim}{M}(\underline{x})\, d\underline{x}\, \underset{\sim}{F}\, \underset{\sim}{G}^{-1} \tag{34}$$

Kelly et al. (Ref 11) obtained the same form of solution for the potential problem by eliminating the source density. For the elastostatics problem this is equivalent to eliminating $\underline{\bar{P}}$ from Equations (25) and (26). With their approach we see that the product of $\underset{\sim}{F}$ and $\underset{\sim}{G}^{-1}$ provides the relation between nodal tractions and displacements. We can write Equation (34) in an abbreviated form as

$$\underset{\sim}{K} = \underset{\sim}{C}\,\underset{\sim}{E} \tag{35}$$

where

$$\underset{\sim}{C} = \int_{\Gamma} \underset{\sim}{N}^{T}(\underline{x})\,\underset{\sim}{M}(\underline{x})\,d\underline{x} \tag{36}$$

and

$$\underset{\sim}{E} = \underset{\sim}{F}\,\underset{\sim}{G}^{-1} \tag{37}$$

Brebbia's direct approach (Ref 21) to obtaining a stiffness matrix for the DBEM also has the same form. For the DBEM however, $\underset{\sim}{E}$ is given by

$$\underset{\sim}{E} = \underset{\sim}{G}^{-1}\,\underset{\sim}{F} \tag{38}$$

where $\underset{\sim}{F}$ is not the same as obtained for the IBEM. The DBEM uses the same fundamental solution; however the roles of the source and field points and the roles of the indices are reversed in obtaining the Somigliana identities.

As previously mentioned, direct formulations of stiffness matrices do not initially yield a symmetric matrix. In addition both the direct and variational formulations produce stiffness matrices which violate equilibrium. Hartman (Ref 23) attributed these problems to the approximations made in obtaining the traction-displacement relationship, Equations (37) and (38), for the IBEM and DBEM, respectively. For the DBEM the equilibrium problem can be addressed when formulating E, Equation (13) is augmented with two additional equations and corresponding Lagrange multipliers, which force the traction distribution to "identically" satisfy equilibrium (Ref 11). Most approaches to these problems modify the stiffness matrix rather than address the source of the problem, the traction-displacement relationship (Ref 20).

Discontinuity of Boundary Element Methods

Regardless of the formulation there is a characteristic of BE formulated stiffness matrices and BE solutions in general which seems to

be ignored in the literature; **BE formulations are inherently incompatible at the interface of homogeneous regions.**

Derivations of BE formulations usually indicate that the governing differential equations are satisfied exactly in the domain while the boundary conditions are only approximately satisfied. Often this is interpreted as meaning the boundary element shape functions constrain the displacement field. Actually the meaning has greater depth. This interpretation is often a consequence of invalid analogies between BEMs and FEMs. The methods are theoretically related and certainly the BEM has borrowed much technology from the FEM; however, the accepted use of the name "boundary element methods" for boundary integral equation techniques is misleading. This interpretation is usually not enlightened by numerical experience since many BE programs can not accurately calculate responses in the near boundary region.

The shape functions used for the so called "boundary elements" do not constrain the displacement field but merely offer an approximation of the boundary values for integration. If they did constrain the displacement field, a single BE region comprised of four linear elements could pass the FE "patch test" (Ref 39); such is not the case. Typically the integral equations are satisfied in a collocation sense (i.e., at discrete points along the boundary) and thus the boundary conditions are satisfied at the collocation points (which coincide with nodal points for continuous element formulations). In between these points, satisfaction of boundary conditions can be greatly in error. This is graphically illustrated in our previous report (Ref 31).

Similar to the displacement-based FEM's satisfaction of equilibrium equations in a nodal sense, the BEM only satisfies boundary conditions in a nodal sense. Many researchers indicate that compatibility between the two methods is easily satisfied by using the same shape functions (Refs 1, 13, 21, 22, and 24); I disagree. However, discontinuity does not imply poor results. On the contrary many incompatible finite elements exhibit improved performance, and many accurate BE solutions have been reported. Recognition of this basic incompatible behavior could be of practical use. It affects the convergence properties of the method and can potentially explain other solution characteristics. One

very useful application could be in mesh refinement. Whether the refinement is a manual or an adaptive process, the variation in computed responses between collocation points can provide a measure on which to base the refinement.

DISCUSSION OF NUMERICAL IMPLEMENTATION

This section discusses the numerical implementation of the IBEM stiffness matrix developed above. Algorithms, in the form of pseudo-code outlines, provide detail on key aspects of the implementation. These algorithms have been implemented in a research code. The numerical studies are incomplete but will be presented in a follow-up report. The scope of this discussion is limited to the calculation of the stiffness matrix; assembly of element stiffness matrices is well documented (Ref 39). Both the IBEM and DBEM are addressed below since their stiffness matrices Equation (35) can have the same form. Rudolphi (Ref 22) provides a general outline for the calculation of a "stiffness" matrix including the BE coefficient matrix calculations. His paper deals principally with the potential problem using the DBEM. Though Rudolphi's work was not referenced for the implementation aspects of this work, the main steps in the calculations are independent of the application and BE formulation.

Due to the complexity of BE and FE software systems emphasis must be placed on the modular design of the coupled software system. In the following discussion I assume BE and FE systems exist and require modification. The stiffness matrix calculation can be organized as three tasks:

(1) Calculation of the coefficient matrices ($\underset{\sim}{G}$ and $\underset{\sim}{F}$) and $\underset{\sim}{C}$

(2) Preliminary $\underset{\sim}{K}$ calculation, inverse and matrix product calculations as shown in Equation (34)

(3) Optional adjustments in $\underset{\sim}{K}$ to satisfy equilibrium and symmetry

The first task requires modification of the BE system. The second and third tasks can be added to the BE system, comprise a separate module, or be incorporated into an element routine of the FE system. In my approach I prefer to "weakly couple" the software systems so that they can still serve as individual research and analysis tools. By this approach the first task is completely performed by the modified BE system. The second and third tasks are performed by a separate program module which except for the three component matrices of $\underset{\sim}{K}$ (i.e., $\underset{\sim}{C}$, $\underset{\sim}{F}$, $\underset{\sim}{G}$,) requires no access to the BE data. An element routine is then added to the FE program which reads data defining the "super-element" connectivity and assembles the BE $\underset{\sim}{K}$ into the global system at the time of assembly. Solution of the FE system provides nodal displacements from which stresses can be calculated in both the FE and BE domains.

When boundary tractions or internal responses are required in the BE region, the displacements associated with BE nodes must be retained. For either BE formulation, boundary tractions can be obtained by the equation: $\bar{t} = \underset{\sim}{E}\ \hat{u}$. For the DBEM the known boundary values in combination with Somigliana's identities (Equation (10)) provide internal responses. For the IBEM the tractions or displacements could be used to solve a boundary value problem for the artificial tractions. Alternatively $\underset{\sim}{G}^{-1}$ could be saved during the $\underset{\sim}{K}$ calculations giving $\bar{P} = \underset{\sim}{G}^{-1}\ \hat{u}$. The effect of the artificial tractions is then integrated to obtain internal responses according to Equations (16) through (18). The remainder of this discussion concentrates on the calculation of the stiffness matrix.

For smaller problems the calculation of the matrices comprising $\underset{\sim}{K}$ dominates the numerical effort. The coefficient matrices are inherently full and their formulation requires extensive numerical integration. The cost of their calculation would appear to increase with the square of the number of boundary elements. There is an element "overhead cost" which follows this trend; however for a program which uses a variable order integration scheme the cost of the average element integration decreases. This is due to the reduction in the order of integration with a decrease in the ratio of element length to collocation point distance (Ref 31 and 32).

Calculation of $\underset{\sim}{E}$ requires an IBEM program to generate both $\underset{\sim}{F}$ and $\underset{\sim}{G}$ for a single homogeneous domain. These are the same equations required to combine homogeneous BE regions by approximate satisfaction of equilibrium and continuity along their interface. Thus many existing BE programs already have this capability. The calculation of two coefficient matrices does not double the numerical effort because much of the overhead in the numerical integrations is common to both types of coefficients.

In the DBEM both $\underset{\sim}{F}$ and $\underset{\sim}{G}$ are always calculated. The only modification which might be required, depending on the program design, is the retention of all coefficients. Some implementations immediately multiply known boundary values by corresponding coefficients and sum the terms into the known vector to form the system of equations (see Equation 15).

Calculating the integration of the shape function product matrix, $\underset{\sim}{C}$, according to Equation (36), is effectively performed at the element level. Because of the locally based nature of the shape functions, $\underset{\sim}{C}$ is very sparse. For quadratic elements, rows associated with a mid-node have three nonzero terms, and those associated with an extreme node have five nonzero terms. In this implementation tractions and displacements are interpolated at the same order (quadratic), and there are no provisions for traction discontinuities. As a result $\underset{\sim}{M}(\underline{x}) = \underset{\sim}{N}(\underline{x})$ and therefore $\underset{\sim}{C}$ is symmetric. Not providing for traction discontinuities does place geometrical constraints on the BE region. However for BE regions comprised of several elements and few corners, Rudolphi (Ref 22) concludes that traction continuity has a negligible effect on the accuracy. For applications where the BEM region is used as an "infinite element" the analyst determines the shape of the FEM/BEM interface and thus geometric discontinuities pose no problem.

For a vector boundary value problem such as the elasticity problem, the two rows of $\underset{\sim}{C}$ corresponding to a common node have identical terms but differ in their position, column. Except for the diagonal terms in $\underset{\sim}{C}$ each term calculated at the element level is complete (i.e., no summation is required) and could be formally assembled into four positions of $\underset{\sim}{C}$. Only the diagonal terms of $\underset{\sim}{C}$ which correspond to shape functions

spanning two elements require the addition of a second term. Thus it is very efficient to both calculate $\underset{\sim}{C}$ and perform its multiplication with $\underset{\sim}{E}$ at an element level. There is a maximum of six unique terms for a single element. Execution and storage requirements associated with $\underset{\sim}{C}$ are trivial compared to $\underset{\sim}{F}$ and $\underset{\sim}{G}$ which are full coefficient matrices.

The algorithm below outlines the calculation of $\underset{\sim}{C}$ at the element level. The research code is written in the Modula-2 language (Ref 40), which is a descendent of Pascal and Modula. Those familiar with Pascal will recognize the dialect below. I have attempted to remove enough syntactical idiosyncrasies from the code to permit communication. Calls to Modula-2 procedures (analogous to FORTRAN subroutines) are supplemented with short descriptions of their purpose. Comments which do not replace actual code are enclosed in (* *)'s. General file operations and variable definitions are omitted. Descriptive variable names supplemented by comments are used to define variables. Many modern languages provide for user defined data types. Nnode and Mnode are variables which can have the enumerated values of (a,c,b), the node indices listed by extreme nodes first. This corresponds to α of Figure 4 taking the values of (1,3,2) respectively. These variables, Nnode and Mnode, serve as nodal indices to arrays and promote internal documentation. More complicated data types such as records have been converted to arrays to simplify the outline.

```
PROCEDURE GenerateShapeProductMatrix
(ContNode,       (* array of continuous element node numbers
                    indexed as (element_number,node_index) *)
 numeltotal,     (* total number of continuous elements *)
 ShapeIntOrder) (* numerical integration order *)

BEGIN (* the GenerateShapeProductMatrix procedure *)
FOR elnum = 1 TO numeltotal DO (* for each element *)
  (* determine shape function (N) & weight-Jacobian product
     values at each integration point *)
  FOR Nnode = a TO b DO (* an index on the node *)
    Write the equation number for node ContNode[elnum,Nnode] to
      the NMfile
  END (* the Nnode loop *)
  FOR intpt = 1 TO ShapeIntOrder DO (* each integration point *)
    position = GAUSSpt[ShapeIntOrder,intpt] (* obtain the
      position of the integration point *)
    SHAPEfunctions(position,Nvalues,intpt) (* calculate the shape
      function values at "position" and save in Nvalues
      indexed as [intpt,node_index] *)
    Calculate the Jacobian at "position"
    WGTdetJ[intpt] = GAUSSwgt[ShapeIntOrder,intpt]*J (* calculate
      the integration weight Jacobian product at each integration
      point *)
  END (* the integration point loop *)
  FOR Nnode = a TO b DO (* for each nonzero shape function *)
    FOR Mnode = Nnode TO b DO (* for each unique combination of
      shape function, i.e., N=M thus symmetry is considered *)
      integral = 0.0 (* initialize integration *)
      FOR intpt = 1 TO ShapeIntOrder DO (* each int. pt *)
        integral = integral + Nvalues[intpt,Nnode]*
          Nvalues[intpt,Mnode]*WGTdetJ[intpt]
      END (* the integration point loop *)
      Write the integral value to the NMfile
    END (* the Mnode, M shape function, loop *)
  END (* the Nnode, N shape function, loop *)
END (* the element loop *)
END (* the GenerateShapeProductMatrix procedure *)
```

The six integration results are written to the NMfile in the order: a-a, a-c, a-b, c-c, c-b, and b-b, where the letters indicate the node index of each shape function in the product.

With all of the component matrices for $\underset{\sim}{K}$ calculated, the second task of calculating an initial nonsymmetrical stiffness matrix can proceed. The third task of satisfying equilibrium and obtaining a symmetric form can be combined with the matrix operations of the second task. The details of integrating the two tasks are highly dependent on the method used to obtain symmetry and equilibrium. For this research code, the tasks are independent, so different methods can be investigated. This also allows the symmetry of the original stiffness matrix to be examined.

The initial steps required in the calculation of $\underset{\sim}{K}$ differ in the two BE formulations. This difference results from the reversal of $\underset{\sim}{F}$ and $\underset{\sim}{G}^{-1}$ in the traction-displacement relations, Equations (37) and (38). As a result of this difference the IBEM is burdened with n^3 additional multiplications and $(n-1)n^2$ additions to obtain $\underset{\sim}{E}$. Additionally the DBEM has reduced memory requirements. $\underset{\sim}{F}$ can be read from disk one column at a time. Each column can immediately be used to calculate the corresponding column of $\underset{\sim}{K}$.

As the order of the system of equations increases, the calculation of the inverse in Equation (37) or (38) becomes the most costly step. For Gauss elimination the cost increases as n^3. A common approach (Ref 22 and 24) to reducing the cost of calculating the inverse is to subdivide the single BE region into several BE regions, providing many small stiffness matrices instead of one large stiffness matrix. This subdivision introduces additional boundary elements, and thus more equations, but the equations corresponding to the complete BE domain are now block banded. Mitsui et al. (Ref 24) suggests that with regard to efficiency there is an optimal degree of subdivision. Subdividing the BE region may also require that traction discontinuities and the violation of equilibrium be treated more rigorously since the occurrence of geometric discontinuities and a relatively small number of elements would be more likely.

For both BE formulations, a column of $\underset{\sim}{E}$ is immediately used to calculate the corresponding column of $\underset{\sim}{K}$. The above algorithm calculates $\underset{\sim}{C}$ at the element level and writes the values to NMfile. An algorithm to calculate an initial $\underset{\sim}{K}$, given the component matrices, is shown below. Values of $\underset{\sim}{C}$ are read from NMfile, consistent with the previous algorithm. In this implementation the pseudo-code given below is a separate program (a Modula-2 program module).

```
MODULE StiffnessMatrix
BEGIN (* the StiffnessMatrix module *)
ReadCmatrix(EqNum,C,numeltotal,order) (* read C: EqNum contains
  the equation numbers for each node indexed as
  [element_number,node_index] and C contains the integration of
  the shape function products indexed as [element_number,0..5] *)
ReadGmatrix (* read G *)
ReadFmatrix (* read F: For the Modula-2 implementation these two
  procedures cause G and F to be read by modules envolved in the
  calculation of E.  This module does not have access to the data
  of G and F. *)

(* Calculate the stiffness matrix, K *)
Initialize K to zero
FOR col = 1 TO order DO (* for each column of K *)
  CalcEcol(col,Ecol) (* calculates a single column of E stored in
    Ecol *)
  FOR elnum = 1 TO numeltotal DO (* for each element *)
    Ccount = 0 (* initialize the index for the C array *)
    FOR node1 = a TO b DO (* node index for the first shape
      function *)
      roweq = EqNum[elnum,node1] (*first row equation for node1*)
      FOR node2 = node1 TO b DO (* for each new combination *)
        coleq = EqNum[elnum,node2] (* first column equation for
          node2 *)
        Cval = C[elnum,Ccount] (* extract the C term from the
          element array *)
        FOR eqincrement = 0 TO 1 DO (* for each nodal dof *)
          Crow = roweq+eqincrement
          Ccol = coleq+eqincrement
          (* calc. contributions in both symmetrical terms *)
          K[Crow,col] = K[Crow,col] + Ecol[Ccol]*Cval
          IF roweq#coleq THEN  (* not a diagonal term *)
            K[Ccol,col] = K[Ccol,col] + Ecol[Crow]*Cval
          END
        END (* the eqincrement loop *)
        increment Ccount
      END (* the node2 loop *)
    END (* the node1 loop *)
  END (* the elnum loop *)
END (* the col loop *)

Adjust K for equilibrium and symmetry
Write the upper triangular portion of K to disk
END (* the StiffnessMatrix module *)
```

As previously noted, due to symmetry and the vector nature of the elasticity problem each element $\underset{\sim}{C}$ value (except diagonal terms) has four positions in the assembled $\underset{\sim}{C}$. The "eqincrement loop" above multiplies each element $\underset{\sim}{C}$ value by four terms (two for diagonal terms) in the column of $\underset{\sim}{E}$ and assembles the products into the corresponding four $\underset{\sim}{K}$ terms. By this approach the sparsity of $\underset{\sim}{C}$ is exploited and the matrix is never assembled.

The last task in calculating a stiffness matrix is adjusting $\underset{\sim}{K}$ to satisfy symmetry and equilibrium. The work of Tullberg and Bolteus (Ref 20) provides guidance, based on numerical studies, for both of these problems. As previously mentioned, numerical studies have shown the nonsymmetrical $\underset{\sim}{K}$ obtained by the direct approach to be more accurate than the symmetrical form obtained by a variational approach. However when combining the matrix with an existing FE system a symmetric form is usually preferable. An exception can occur if the FEM is used to model plasticity governed by nonassociative flow rule; in this case the resulting FE system can be nonsymmetric. In this study the stiffness matrix is symmetrized by averaging the off-diagonal terms.

In the initial implementation of the research code, equilibrium considerations have not been included. Our principal interest is in infinite domain problems where previously mentioned techniques are not applicable. For finite domain problems, one of the methods implemented by Tullberg and Bolteus (Ref 20) to solve the equilibrium problem might be improved upon. The equilibrium error for a given direction was corrected by adding the average error to each term. This approach did not consider the relative magnitude of each term. A better approximation might be obtained by basing the error distribution on the normalized magnitude of each term where the sum of absolute values of the stiffness terms is the normalizing factor.

SUMMARY AND CONCLUSIONS

Fundamental differences in the formulations of the finite element and boundary element methods result in corresponding strengths and weaknesses. A combination of the methods using the strengths of each may allow some classes of problems to be solved more effectively. Applications with infinite or semi-infinite domains and fracture mechanics problems are candidates for combined solution approaches.

The DBEM and IBEM are the most common integral equation methods in use. They both have their origins in classical potential theory and the fundamental singular solution is a key component of their derivations. For both methods the approximations common to most formulations are: (1) integrating in a piecewise manner, and (2) solving the integral equations in a collocation sense. The collocation approximation limits the satisfaction of boundary conditions to discrete points; for coupled solutions interface boundaries will be inherently incompatible.

Coupling approaches can be categorized based on several parameters. The most basic classification indicates which numerical method hosts the coupling (i.e., is the final form of the equations "BE like" or "FE like"). This study addresses FE-hosted or stiffness couplings which can be further categorized by the derivation of the stiffness matrix. Either the stiffness matrix is obtained directly or via a variational statement of the problem. In both cases the key matrices are the fully populated coefficient matrices generated by the BEMs and a sparse matrix obtained by the integration of shape function products. The BEM coefficient matrices are manipulated to establish a relationship between the boundary tractions and displacements; this relation is then used directly to obtain a stiffness matrix or substituted into a boundary variational statement.

The two major problems with the BE stiffness matrix are its lack of symmetry and violation of equilibrium. A variational formulation inherently produces a symmetric system which is equivalent to taking the symmetric component of the corresponding directly-formulated stiffness

matrix. Investigators have offered several alternatives for dealing with the equilibrium problem ranging from the introduction of Lagrange multipliers which enforce equilibrium to ad hoc adjustments of the stiffness matrix.

In most cases, stiffness matrix calculations for both BE formulations follow the same main steps. However, the direct BEM is better suited to coupled solution approaches because of its simpler traction-displacement relationship. For smaller problems the calculation of matrices comprising $\underset{\sim}{K}$ require the most numerical effort. For larger problems the inverse calculation of a fully populated matrix dominates the numerical effort. The only known method for reducing this cost is to subdivide the homogeneous BE region into a number of smaller regions--trading the calculation of a large inverse for the calculation of several smaller ones.

The only matrix in the stiffness calculation not calculated by standard BE programs is the shape function product matrix $\underset{\sim}{C}$. For continuous BEs $\underset{\sim}{C}$ is symmetric. Additionally the locally based nature of the shape functions makes $\underset{\sim}{C}$ very sparse. The calculation of $\underset{\sim}{C}$ and its multiplication are effectively performed at the element level.

RECOMMENDATIONS

There are several areas of investigation which could improve our fundamental understanding of coupled solution techniques or improve their numerical effectiveness. There are also off-shoot areas from this and previous investigations which are not within the scope of this project. Additionally there are areas which must be investigated to attain our long term objective but could require significant implementation effort. These later areas are separately included below but would require 6.2 level funding. For FY87, the following areas are of interest:

- numerical comparison of BEM, FEM, stiffness-BEM, and coupled solutions
- numerical study of incompatibility

  - -- between FEM and IBEM using the same shape functions
  - -- inherent to the DBEM
- theoretical basis and numerical techniques for dynamic problems with infinite domains
- theoretical basis and numerical techniques for semi-infinite domains.

The first two areas are the recommended core for the FY87 investigation. The later two are rather general and could be investigated as time permits. Areas of investigation which either are 6.2 level efforts or require significant 6.2 support include:

- numerical study of coupled solution behavior for static problems
- symmetry considerations in formulating the BE equations
- further investigation of element integrations
  - -- special techniques for integrations over singularities
  - -- special techniques for the near boundary region
  - -- order of integration calculation
- Galerkin formulation of BEMs
- development of a DBEM code to support 6.1 efforts
- visco-elasticity.

Two topics which are "off-shoots" of this and previous investigations are:

- adaptive mesh refinement based on boundary condition violation
- BE formulated crack-tip "elements" in FE hosted couplings using the recursive integration technique.

We appear to have unique aspects in our approach to both of these problems.

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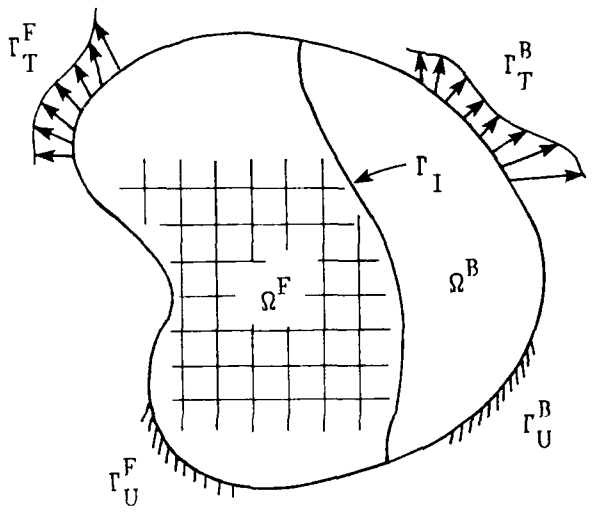

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$$\Gamma_U = \Gamma_U^F \cup \Gamma_U^B$$

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Figure 1. Two-dimensional elastostatics problem using coupled solution approach.

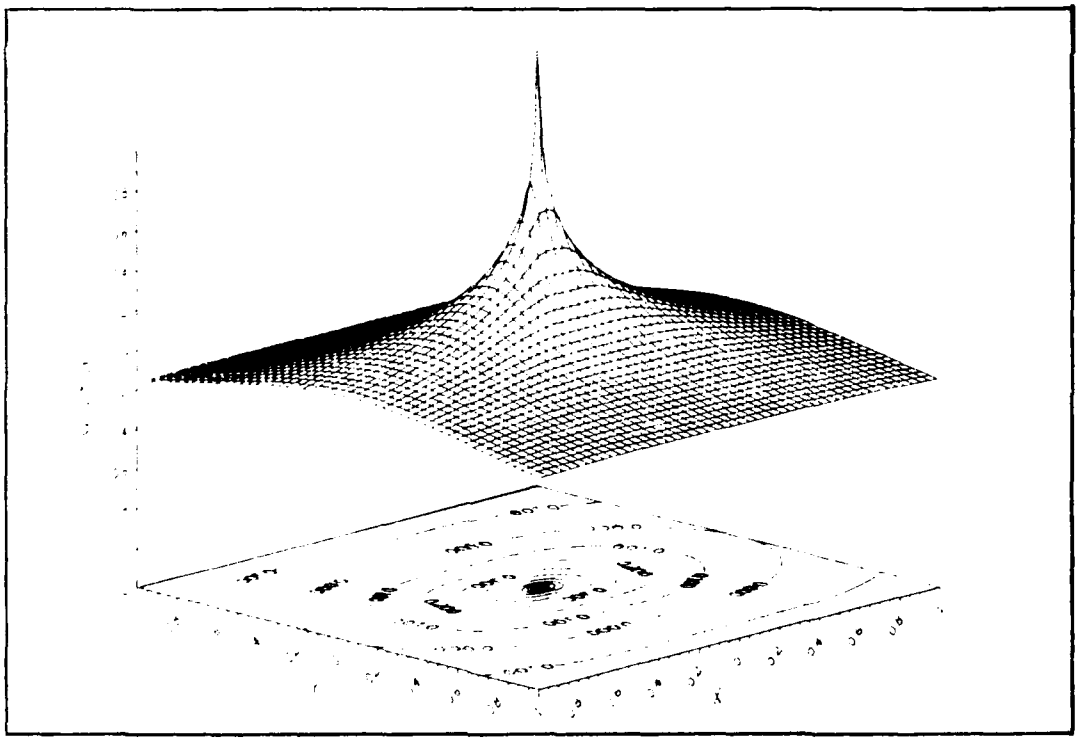

Figure 2. Displacement fundamental solution, G_{11}.

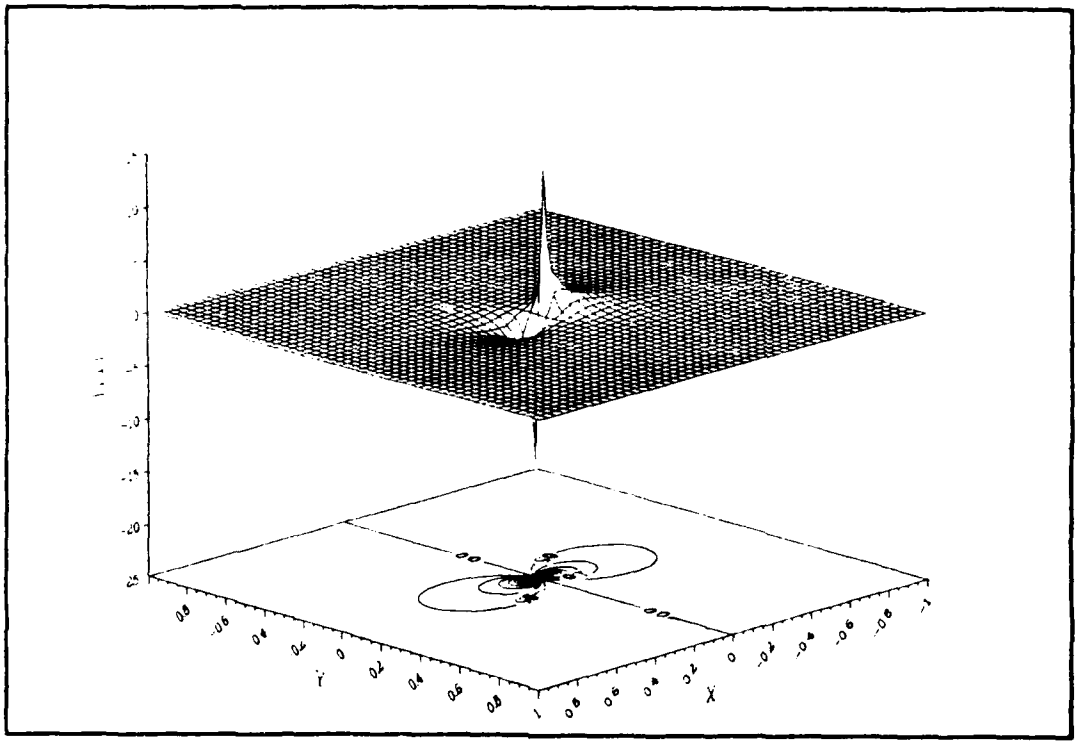

Figure 3. Stress fundamental solution, T_{111}.

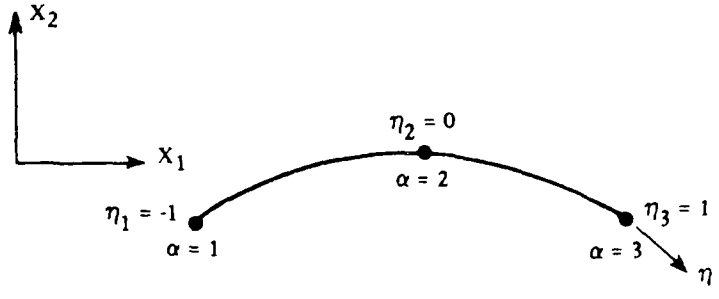

Figure 4. Quadratic isoparametric boundary element.

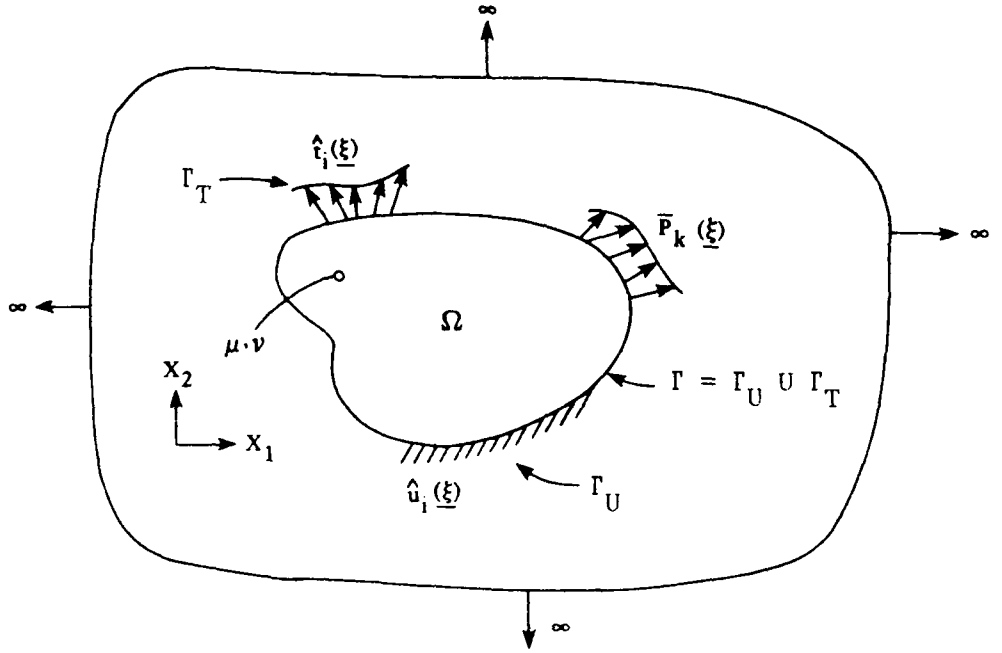

Figure 5. Two-dimensional elastostatics problem embedded in an infinite plane.

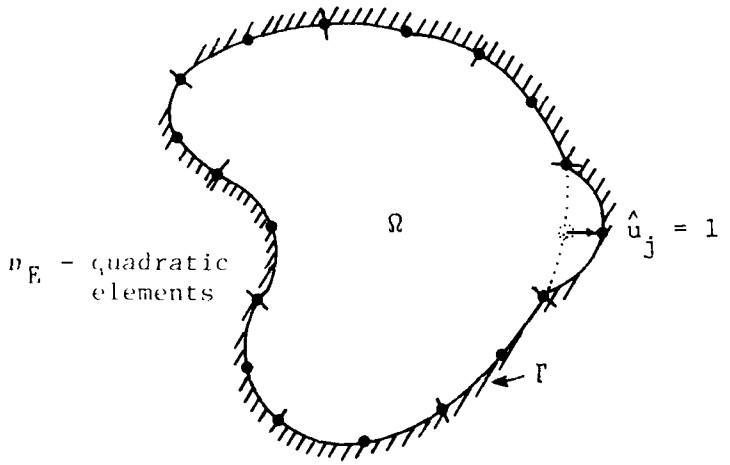

Figure 6. Unit perturbation of a BEM region's j^{th} degree of freedom.

Appendix A

LIST OF SYMBOLS

The most general form of a variable (i.e., with the least amount of supplementary symbols) is defined. For example, u_i represents a displacement vector but depending on the notation it may be continuous or discrete, known or unknown. There are some non-unique uses of Latin symbols. For these cases supplementary symbols are included as necessary. Indicial notation is only used for vector and tensor quantities. The indice range in this case is two.

Mathematical Symbols

$V(x)$	Explicitly defines the quantity as continuous as opposed to discrete.
$\underset{\sim}{V}$	A rectangular or square matrix of functions (e.g., $\underset{\sim}{M}(x)$) or constants (e.g., $\underset{\sim}{K}$).
$\underline{V}$	A column vector.
$\hat{V}$	A known value(s); for example, $\hat{\underline{u}}$ is a vector of known nodal displacements.
$\bar{V}$	An unknown value(s); for example, $\bar{\underline{t}}$ is a vector of unknown nodal tractions.
$\underset{\sim}{V}^T$	Transpose of a matrix or vector. $\underline{V}^T$ is a row vector.
$\underset{\sim}{V}^{-1}$	Inverse of a matrix.

Latin Symbols

A_{ik}	A constant tensor which provides a zero displacement for the fundamental singular solution at a reference distance r_o. $A_{ik} = -\delta_{ik} C_1[C_2 \ln(r_o)-1]$.
$\underset{\sim}{C}$	Square, sparse matrix which contains boundary integrations of traction and displacement shape function products.

C1, C2, C3, C4	Constants used in defining the fundamental solution. Functions of the material constants.
$e_k(\xi)$	A unit concentrated force used in the definition of the fundamental solution.
E	Young's modulus.
$\underset{\sim}{E}$	Square matrix relating nodal tractions to nodal displacements, $\underline{t} = \underset{\sim}{E}\,\underline{u}$
$E_{ijk}(\underline{x},\underline{\xi})$	Kernal function in Somigliana's 2'nd identity relating boundary displacements to the stress field.
$\underline{f}$	Column vector of generalized nodal forces.
$F_{ij}(\underline{x},\underline{\xi})$	Fundamental singular solution, Kelvin solution, for traction.
$\underset{\sim}{F}$	A matrix of element integrations involving the traction fundamental solution. Both the direct and indirect formulations consist of $\underset{\sim}{F}$ matrices. The two $\underset{\sim}{F}$ matrices differ due to the switch in roles of the field and source point and the indices.
$G_{ij}(\underline{x},\underline{\xi})$	Fundamental singular solution, Kelvin solution, for displacement.
$\underset{\sim}{G}$	A matrix of element integrations involving the displacement Kelvin solution. Due to the symmetries in the displacement Kelvin solution the $\underset{\sim}{G}$ matrices for the direct and indirect formulations are equivalent.
$H_{ijk}(\underline{x},\underline{\xi})$	Kernal function in Somigliana's 2'nd identity relating boundary tractions and body forces to the stress field.
$\underset{\sim}{M}(\underline{x})$	Traction shape functions used with nodal traction values to approximate a piecewise continuous traction distribution.
$\underset{\sim}{N}(\underline{x})$	Displacement shape functions used with nodal displacement values to approximate a continuous displacement distribution.
n	Order of the system of equations.
$\bar{P}_i$	Artificial boundary traction vector, analogous to simple-layer source or source density of potential problems.

r	Distance from the field to the source point. $r^2 = y_i\ y_i$.
t_i	Actual boundary traction vector.
$T_{ijk}(\underline{x},\underline{\xi})$	Fundamental singular solution, Kelvin solution, for stress.
u_i	Displacement vector.
x_i	Position vector to a point on the boundary or within the domain of the problem.
y_i	Vector from the source to the field point. $y_i = x_i - \xi_i$.
z_i	Position vector used in the integration of body forces.

Greek Symbols

Γ	Complete, finite boundary of the problem.
δ_{ij}	Kronecker delta symbol.
ε_{ij}	Strain tensor.
λ	A Lame's constant.
μ	A Lame's constant, the shear modulus.
ν	Poisson's ratio.
ξ_i	Position vector to a point on the boundary of the problem.
σ_{ij}	Stress tensor.
Ω	Domain of the problem.
ψ_i	Vector of body forces.

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DATE
9-88
DTIC